
VI. *A Demonſtration of the* 11*th Propoſition of Sir* Iſaac Newton*'s Treatiſe of* Quadratures, *By Mr.* Benjamin Robins.

THIS Propoſition conſiſts of two Parts: The firſt is as follows.

Let there be any Curve A D I, whoſe Abſciſſe A B ſhall be denoted by z, and its Ordinate B D by y; which may be related in any manner to the Abſciſſe. And calling this the firſt Curve, let other Curves AEK, AFL, AGM, AHN, *&c.* be formed to the common Abſciſſe A B, or z, by making the Ordinate B E of the ſecond Curve always equal to the Area A B D of the firſt divided by Unity; the Ordinate B F of the third equal to the Area A B E of the ſecond divided by Unity; the Ordinate B G of the fourth equal to the Area A B F of the third divided by Unity; and ſo on continually. Suppoſe now, that other Curves AOS, A P T, A Q V, A R W, be deſcribed to the ſame common Abſciſſe A B or z; in which Curves the Ordinate B O of the Curve A O S ſhall be equal to $z y$, the Ordinate B P of the Curve A P T equal $z^2 y$, the Ordinate B Q of the Curve A Q V equal to $z^3 y$, the Ordinate B R of the Curve A R W equal to $z^4 y$, *&c.* And let the whole Area A C I be denoted by A, the Area A C S by B, the Area A C T by C, the Area A C V by D, the Area A C W by E, *&c.* Then the Series of Curves ADI, A E K, A F L, A G M, A H N are thus meaſured:

The

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to *Philosophical Transactions (1683-1775).* www.jstor.org

The Area of the firſt Curve ADI is $=A$
——of the ſecond AEK is $= zA - B$
——of the third AFL $= \frac{zzA - 2zB + C}{2}$
——of the fourth AGM $= \frac{z^3A - 3z^2B + 3zC + D}{6}$
——of the fifth AHN $= \frac{z^4A - 4z^3B + 6z^2C - 4zD + E}{24}$

and ſo on perpetually. Here in all the Curves following the firſt, the Index of the higheſt Power of z is always the Number which expreſſes the Diſtance of the Curve from the firſt, and afterwards decreaſes regularly by Unity; the firſt Term is multiplied into A, the ſecond into B, the third into C, the fourth into D, and ſo on; the Coefficients are the ſame as in a Binomial raiſed to the higheſt Power of z, and the Diviſor is ſo many Terms of this Progreſſion $1 \times 2 \times 3 \times 4 \times 5 \times 6$ &c. as is expreſs'd by a Number equal to the higheſt Index of z. Otherwiſe ſuppoſing n to repreſent the Diſtance of the Curve to be meaſured from the firſt; then the Area ſought will be found by extending $\overline{z-1}|^n$ into a Series, and multiplying the firſt Term by A, the ſecond by B, the third by C, the fourth by D, &c. and dividing the whole by $n \times \overline{n-1} \times \overline{n-2}$ &c. continued to Unity.

SECOND PART.

Suppoſing the firſt, ſecond, third, &c. Curves to be the ſame as before: Let t denote the whole Abſciſſe AC, and put x for BC: Then deſcribe the Curves CXA, CYA, CZA, CΓA, where BX ſhall be equal to xy, BY $=x^2y$, BZ $=x^3y$, BΓ $=x^4y$, &c. This being done, and in the Series of Curves CIDA, CXA,

CXA, CYA, CZA, CΓA, *&c.* the firſt Area CIDA being put equal to P, the ſecond CXA equal to Q, the third CYA=R, the fourth CZA=S, the fifth CTA=T, *&c.* the whole Areas of the aforeſaid Series of Curves are alſo determin'd as follows.

The firſt AIC=P
The ſecond AKC=Q
The third ALC=$\frac{1}{2}$ R
The fourth AMC=$\frac{1}{6}$ S
The fifth ANC=$\frac{1}{24}$ T.

Here the Area's P,Q,R,S,T are divided by Numbers produced by multiplying as many Terms of this Series $1 \times 2 \times 3 \times 4 \times 5$ *&c.* together, as in the former Caſe.

Demonſtration of the Firſt Part.

Let the Area ABD be denoted by *a*, the Area ABO by *b*, ABP by *c*, ABQ by *d*, and ABK by *e*. Then it is evident, that

The Fluxion of the Area ABD is$=\dot{z} \times BD = \dot{z}y = \dot{a}$
The Fluxion of the Area ABO is$=\dot{z} \times BO = \dot{z}zy = \dot{b}$
The Fluxion of the Area ABP is$=\dot{z} \times BP = \dot{z}z^2y = \dot{c}$
&c. &c.

Hence

$$z \times \dot{a} \text{ is } (= \dot{z}zy) = \dot{b}$$

$$z^2 \times \dot{a} \text{ is } (= \dot{z}z^2y) = z\dot{b} = \dot{c}$$

$$z^3 \times \dot{a} \text{ is } (= \dot{z}z^3y) = z^2\dot{b} = z\dot{c} = \dot{d}.$$

Or generally,

$$z^n \times \dot{a} = z^{n-1} \times \dot{b} = z^{n-2} \times \dot{c} = z^{n-3} \times \dot{d}, \&c.$$

Now

Now as $\dot{z} \times$ ABD or $\dot{z} \times a$ is $=$ Fluxion of ABE, if you add to the firſt Part $z \times \dot{a}$ $(= \dot{z} z y)$ and its Equal $\dot{b}$ to the other Part, it follows, that

$$\left.\begin{array}{r} \dot{z} \times a \\ + z \times \dot{a} \end{array}\right\} = \text{Fluxion of ABE} + \dot{b}$$

And taking the Fluents
$z \times a = \text{ABE} + b$ or $\text{ABE} = z \times a - b$; and when z or AB becomes $=$ AC, then ABE becomes ACK, and a and b become A and B; therefore ACK is $= z \times \text{A} - \text{B}$.

Again, The Ordinate BF of the next Curve is equal to ABE, which has been proved equal to $z \times a - b$. Conſequently the Fluxion of ABF is $= \dot{z} z a - \dot{z} b$; and if you add to the firſt Part of this Equation $\frac{1}{2} z^2 \times \dot{a} - \dot{z} b$ $(= \frac{1}{2} z^2 \dot{z} y - \dot{z} z^2 y = - \frac{1}{2} \dot{z} z^2 y)$ and its Equal $- \frac{1}{2} \dot{c}$ on the other, it follows, that

$$\left.\begin{array}{r} \dot{z} z a - \dot{z} b \\ + \frac{1}{2} z z \dot{a} - z \dot{b} \end{array}\right\} = \text{Fluxion of ABF} - \frac{1}{2} \dot{c}.$$

And taking the Fluents
$\frac{1}{2} z^2 a - z b = \text{ABF} - \frac{1}{2} c$; or by tranſpoſing
$\text{ABF} = \frac{z^2 a - 2 z b + c}{2}$; or ſuppoſing z equal to AC,
$\text{ACL} = \frac{z^2 \text{A} - 2 z \text{B} + \text{C}}{2}$.

The Ordinate BG is equal to ABF, which has been proved equal to $\frac{z^2 a - 2 z b + c}{2}$: Therefore the Fluxion of ABG is equal to $\frac{\dot{z} z^2 a - 2 \dot{z} z b + \dot{z} c}{2}$

And

And adding $\frac{1}{6} z^3 a - \frac{1}{2} z^2 \dot{b} + \frac{1}{2} z \dot{c}$ $\left(= \frac{1}{6} \dot{z} z^3 y - \frac{1}{2} \dot{z} z^3 y + \frac{1}{2} \dot{z} z^3 y = \frac{1}{6} \dot{z} z^3 y\right)$ on one ſide, and its Equal $\frac{1}{6} \dot{d}$ on the other, it will be

$$\left.\begin{array}{l} \frac{1}{2} \dot{z} z^2 a - \dot{z} z b + \frac{1}{2} \dot{z} c \\ + \frac{1}{6} z^3 \dot{a} - \frac{1}{2} z^2 \dot{b} + \frac{1}{2} z \dot{c} \end{array}\right\} = \text{Fluxion of } ABG + \frac{1}{6} \dot{d}$$

And taking the Fluents
$\frac{1}{6} z^3 a - \frac{1}{2} z^2 b + \frac{1}{2} z c = ABG + \frac{1}{6} d$; and tranſpoſing, $ABG = \frac{z^3 a - 3 z^2 b + 3 z c - d}{6}$; or ſuppoſing z equal to AC; then $ACM = \frac{z^3 A - 3 z^2 B + 3 z C - D}{6}$.

In the ſame manner the Fluxion of *ABH* is equal to $\frac{\dot{z} z^3 a - 3 \dot{z} z^2 b + 3 \dot{z} z c - \dot{z} d}{6}$; and adding on one ſide $\frac{1}{24} z^4 \dot{a} - \frac{1}{6} z^3 \dot{b} + \frac{1}{4} z^2 \dot{c} - \frac{1}{6} z \dot{d}$, and its Equal $- \frac{1}{24} \dot{e}$ on the other, it becomes

$$\left.\begin{array}{l} \frac{1}{6} \dot{z} z^3 a - \frac{3}{6} \dot{z} z^2 b + \frac{3}{6} \dot{z} z c - \frac{1}{6} \dot{z} d \\ + \frac{1}{24} z^4 \dot{a} - \frac{1}{6} z^3 \dot{b} + \frac{1}{4} z^2 \dot{c} - \frac{1}{6} z \dot{d} \end{array}\right\} = \text{Fluxion of } ABH - \frac{1}{24} \dot{e}$$

And taking the Fluents
$\frac{1}{24} z^4 a - \frac{1}{6} z^3 b + \frac{1}{4} z^2 c - \frac{1}{6} z b = ABH - \frac{1}{24} e$: therefore $ABH = \frac{z^4 a - 4 z^3 b + 6 z^2 c - 4 z d + e}{24}$;
Or $ACN = z^4 A - 4 z^3 B + 6 z^2 C - 4 z D + E$, ſuppoſing z equal to AC. In like manner you may proceed to meaſure any of theſe Curves: and you will always find their Value the ſame as is expreſſed in the Propoſition.

De-

Demonſtration of the Second Part.

Suppoſe any Curve whoſe Diſtance from the firſt is denoted by n; then the Curve whoſe Abſciſſe is BC or x, and its Ordinate $x^n y$ divided by $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}$ &c. continu'd to Unity will be equal to it, when x is equal to AC or t.

It is evident, that when the Areas ABD, ABO, ABP, ABQ, ABR, &c. decreaſe, the Areas BCID, BCSO, BCTP, BCVQ, BCWR increaſe reſpectively; and conſequently the Decrements of the Areas ABD, ABO, ABP, &c. or their Fluxions with a negative Sign, are the Increments or Fluxions of the Areas BCID, BCSO, BCTP, &c. that is, calling the Area BCID, α; the Area BCSO, β; the Area BCTP, γ; BCVQ, δ; BCWR, ε: then $\dot{\alpha}=-\dot{a}$, $\dot{\beta}=-\dot{b}$, $\dot{\gamma}=-\dot{c}$, $\dot{\delta}=-\dot{d}$, $\dot{\varepsilon}=-\dot{e}$.

Now the Fluxion of the Curve, whoſe Abſciſſe is $=x$, or BC, and its Ordinate $=x^n y$ is $\dot{x}\,x^n y$; that is, equal to $\dot{x}\,y\times\overline{t-z}|^n$; x being $=t-z$; or ſince the Increment of x, or $\dot{x}$ is equal to the Decrement of z, or $-\dot{z}$, the Fluxion of the ſame Curve is equal to $-\dot{z}\,y\times\overline{t-z}|^n=-\dot{z}\,y$ in $t^n-n\times t^{n-1}z+n\times\frac{n-1}{2}t^{n-2}\times z^2$ &c. $=-t^n\dot{z}y+nt^{n-1}\dot{z}zy-n\times\frac{n-1}{2}t^{n-2}\dot{z}z^2y$, &c.

that is, $=t^n\times-\dot{a}-nt^{n-1}\times-\dot{b}+n\times\frac{n-1}{2}t^{n-2}\times-\dot{c}$, &c.

or $=t^n\dot{\alpha}-nt^{n-1}\beta+n\times\frac{n-1}{2}t^{n-2}\dot{\gamma}$, &c. and taking the Fluents, the Area of the Curve, whoſe Abſciſſe is x, or BC, and Ordinate $x^n y$, is equal to $t^n\alpha-nt^{n-1}\beta+n\times\frac{n-1}{2}t^{n-2}\gamma$ &c. But when x is equal

to

to AC, then α, β, γ, &c. will be equal to A, B, C, &c. as is very evident; conſequently the Area of the Curve whoſe Abſciſſe is x, and Ordinate $x^n y$, when x is $=$ AC, is $t^n A - nt^{n-1} B + n \times \frac{n-1}{2} \times t^{n-2} C$ &c. that is equal to $\overline{t-1}|^n$ thrown into a Series, and the firſt Term multiplied by A, the ſecond by B, the third by C, &c But $\overline{t-1}|^n$ thrown into a Series, and the firſt Term multiplied by A, the ſecond by B, the third by C, &c, and then the whole divided by $n \times \overline{n-1} \times \overline{n-2}$, &c. continued to Unity, is equal to the Area of the Curve, whoſe Place in the Series is denoted by n: Therefore the Area of the Curve, whoſe Abſciſſe is equal to x, and its Ordinate to $x^n y$, taken when x is equal to AC, and divided by $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. continued to Unity, is equal to the Area of a Curve whoſe Place in the Series is denoted by n; that is, Q, which is the Area of a Curve, whoſe Abſciſſe is x, and Ordinate $x y$ taken when x is $=$ AC, is equal to the ſecond Curve AKC; half R, which is the Area to the Abſciſſe x, and Ordinate $x^2 y$, taken in the ſame manner, is equal to the third Curve ALC; $\frac{1}{6}$S, which is a like Area to x and $x^3 y$, is equal to the fourth Curve AMC; $\frac{1}{24}$T, the Area to x and $x^4 y$, x being equal to AC, is equal to the fifth Curve ANC; and ſo on perpetually Q.E.D.

VII. *Muris*

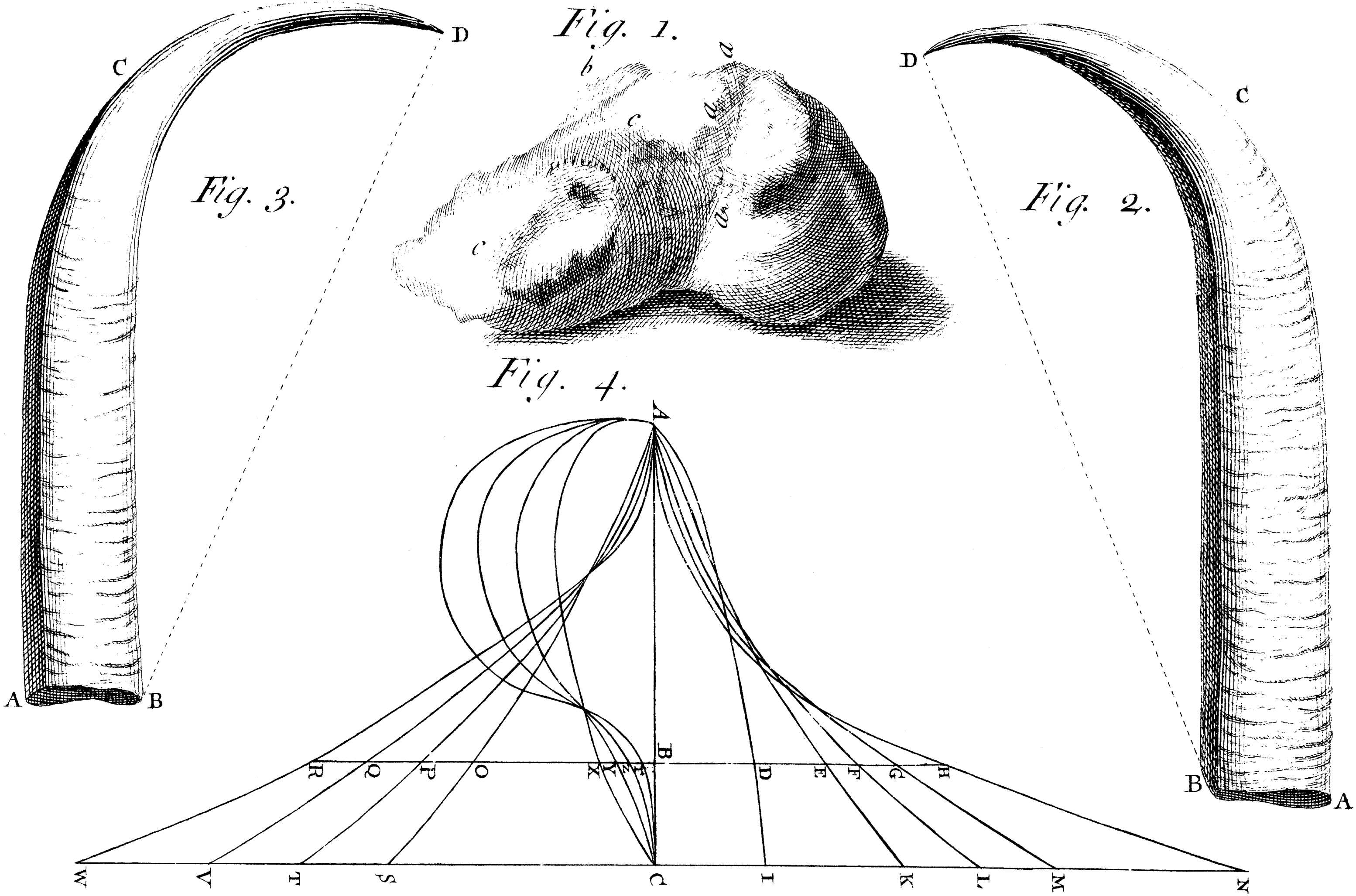

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.